Structure Ranks of Matrices

Miroslav Fiedler
Mathematics Institute
Czechoslovak Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic
and
IMA / University of Minnesota
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

Submitted by Richard A. Brualdi

ABSTRACT

The structure rank of a matrix, i.e. the maximum order of a nonsingular submatrix all of whose entries are located in a given structure (a subset of $M \times N$, where M, N are the row- and column-index sets, respectively) were previously studied for the off-diagonal part. Here, results on more general ranks, e.g. off-block-diagonal, strictly upper-triangular, and strictly upper-block-triangular, are presented. In particular, it is shown that for a nonsingular matrix A, in each of the last-mentioned three cases, the ranks of A and A^{-1} coincide.

1. INTRODUCTION

In [2], the notion of the off-diagonal rank of a square matrix was introduced as the maximum order of a nonsingular submatrix of A none of whose entries is a diagonal entry of A. In particular, it was shown that the off-diagonal ranks of a nonsingular matrix and its inverse coincide.

We intend to generalize this rank and obtain results of a similar nature. As usual, for an $m \times n$ matrix A, $M = \{1, 2, \dots, m\}$, $N = \{1, 2, \dots, n\}$, and $\alpha \subseteqq M$, $\beta \subseteqq N$, we denote by $A[\alpha | \beta]$ the submatrix of A with row indices in α and column indices in β.

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Under a *structure* S in the class of all such $m \times n$ matrices we understand a subset of $M \times N$. We then define the *S-rank* of A, denoted $r(S; A)$, as the maximum order of a nonsingular submatrix of A all of whose entries are in S; in other words,

$$r(S; A) = \max\{\text{rank } A[\alpha|\beta] \mid \alpha \times \beta \subseteqq S\}. \tag{1}$$

Thus, for square $n \times n$ matrices and $S_\sigma = N \times N \setminus \{(1,1), (2,2), \ldots, (n,n)\}$, $r(S_\sigma; A)$ is the off-diagonal rank of A.

We shall also say that a structure S has *combinatorial rank* $\text{cr}(S)$ if

$$\text{cr}(S) = \min\{k \mid P \subseteqq M, Q \subseteqq N, S \subseteqq (P \times N) \cup (M \times Q), |P| + |Q| = k\}.$$

The following theorem [1, Corollary 3] will be of crucial importance in the sequel:

THEOREM A. *Let A be a nonsingular $n \times n$ matrix, and let α, β be subsets of N. Then*

$$\text{rank } A^{-1}[\alpha|\beta] = \text{rank } A[N \setminus \beta | N \setminus \alpha] + |\alpha| + |\beta| - n. \tag{2}$$

COROLLARY B. *For a nonsingular $n \times n$ matrix A and $\alpha \subseteqq N$,*

$$\text{rank } A^{-1}[\alpha | N \setminus \alpha] = \text{rank } A[\alpha | N \setminus \alpha]. \tag{3}$$

2. RESULTS

The following is immediate:

THEOREM 1. *Let S_1, S_2 be structures on $M \times N$ such that $S_1 \subseteqq S_2$. Then for each $|M| \times |N|$ matrix A,*

$$r(S_1; A) \leqslant r(S_2; A) \leqslant r(S_1; A) + \text{cr}(S_2 \setminus S_1).$$

Let us now generalize the theorem on the off-diagonal rank mentioned in the Introduction to off-block-diagonal ranks:

THEOREM 2. *Let $N = N_1 \cup N_2 \cup \cdots \cup N_p$ be a decomposition (i.e., $N_i \cap N_j = \emptyset$ for all i, j, $i \neq j$) of N, $|N| = n$. Let*

$$S = N \times N \setminus \bigcup_{i=1}^{p} (N_i \times N_i).$$

Then for every nonsingular $n \times n$ matrix A,

$$r(S; A^{-1}) = r(S; A). \tag{4}$$

Proof. Since (4) is symmetric with respect to inversion, assume that

$$r(S; A^{-1}) \geqslant r(S; A).$$

Let $r(S; A^{-1}) = \operatorname{rank} A^{-1}[\alpha_0 | \beta_0]$, $\alpha_0 \times \beta_0 \subseteqq S$, let $I = \{i \mid N_i \cap \alpha_0 \neq \emptyset\}$, and let

$$\alpha_1 = \bigcup_{i \in I} N_i.$$

Since $\beta_0 \cap \alpha_1 = \emptyset$, we have $\beta_0 \subseteqq N \setminus \alpha_1$, so that

$$\alpha_0 \times \beta_0 \subseteqq \alpha_1 \times (N \setminus \alpha_1) \subseteqq S.$$

By Theorem 1 and Corollary B,

$$\begin{aligned} r(S; A^{-1}) &\leqslant \operatorname{rank} A^{-1}[\alpha_1 | N \setminus \alpha_1] \\ &= \operatorname{rank} A[\alpha_1 | N \setminus \alpha_1] \\ &\leqslant r(S; A). \end{aligned}$$

Thus (4) follows. ∎

In the next theorem, we prove a similar statement for strictly block-triangular structures.

THEOREM 3. *Let $N = N_1 \cup N_2 \cup \cdots \cup N_p$ be a decomposition of N; let*

$$S_t = \bigcup_{(i,j),\, i<j} (N_i \times N_j).$$

Then, for any nonsingular $n \times n$ matrix A,

$$r(S_t; A^{-1}) = r(S_t; A). \tag{5}$$

Proof. We shall first prove a lemma.

LEMMA. *In our notation,*

$$r(S_t;A) = \max_{q=1,2,\dots,p} \operatorname{rank} A\left[\bigcup_{i=1}^{q} N_i \,\middle|\, N\setminus \bigcup_{i=1}^{q} N_i\right]. \tag{6}$$

Proof. Denote the right-hand side in (6) by R; set

$$\sigma_q = \bigcup_{i=1}^{q} N_i \times \left(N \setminus \bigcup_{i=1}^{1} N_i\right), \qquad q=1,\dots,p.$$

Since $\sigma_q \subseteqq S_t$ for each q, it follows that

$$r(S_t;A) \geqslant r(\sigma_q;A) \qquad \text{for each } q,$$

so that

$$r(S_t;A) \geqslant R.$$

Let

$$r(S_t;A) = \operatorname{rank} A[\alpha_0|\beta_0], \qquad \alpha_0 \times \beta_0 \subseteqq S_t.$$

Define $I=\{i \mid \alpha_0 \cap N_i \neq \emptyset\}$, $J=\{j \mid \beta_0 \cap N_j \neq \emptyset\}$, $\alpha_1 = \bigcup_{i\in I} N_i$. Since $\alpha_0 \times \beta_0 \subseteqq S_t$, every index in I is smaller than every index in J. We have thus

$$\alpha_0 \times \beta_0 \subseteqq \alpha_1 \times (N\setminus \alpha_1),$$

so that

$$\begin{aligned} r(S_t;A) &\leqslant \operatorname{rank} A[\alpha_1|N\setminus\alpha_1] \\ &\leqslant R. \end{aligned}$$ ∎

To return to the proof of Theorem 3, (5) follows immediately, since by Corollary B, the right-hand sides of (6) for A and A^{-1} are equal. ∎

We can thus speak about the *strictly upper-triangular rank* of A in the case that

$$S_u = \bigcup_{1\leqslant j<k\leqslant n} (j,k), \tag{7}$$

and about the *strictly upper-block-triangular rank* of A if

$$S_{ub} = \bigcup_{1 \leqslant j < k \leqslant p} (N_j \times N_k) \tag{8}$$

and all indices in N_k are smaller than all indices in N_{k+1}, $1 \leqslant k \leqslant p-1$. We have thus

COROLLARY 4. *If A is a nonsingular $n \times n$ matrix, then the strictly upper-triangular ranks of A and A^{-1} coincide. The same is true for strictly upper-block-triangular ranks.*

In the next theorem, we admit into S_u in (7) the diagonal entries as well, thus obtaining the *weakly upper-triangular rank*, or S_w-rank.

THEOREM 5. *The weakly upper-triangular ranks of A and A^{-1} differ at most by one*:

$$\left| r(S_w; A) - r\left(S_w; A^{-1}\right)\right| \leqslant 1. \tag{9}$$

Proof. As in the proof of the Lemma, one shows easily that

$$r(S_w; A) = \max_{k=1,\dots,n} \operatorname{rank} A\big[\{1,\dots,k\}|\{k,k+1,\dots,n\}\big]. \tag{10}$$

By Theorem A,

$$\begin{aligned} &\operatorname{rank} A\big[\{1,\dots,k\}|\{k,\dots,n\}\big] \\ &\qquad = \operatorname{rank} A^{-1}\big[\{1,\dots,k-1\}|\{k+1,\dots,n\}\big] + 1 \end{aligned}$$

(and just 1 on the right-hand side if $k=1$ or $k=n$). Hence

$$\begin{aligned} &\operatorname{rank} A^{-1}\big[\{1,\dots,k\}|\{k,\dots,n\}\big] \\ &\qquad \leqslant \operatorname{rank} A^{-1}\big[\{1,\dots,k-1\}|\{k+1,\dots,n\}\big] + 2 \\ &\qquad = \operatorname{rank} A\big[\{1,\dots,k\}|\{k,\dots,n\}\big] + 1. \end{aligned}$$

Thus also the maxima differ at most by one, which, by (10), yields (9). ∎

EXAMPLE. For

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

one has

$$A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix},$$

so that equality is attained in (9).

In the following theorem, the relation of the ranks to the *LU* decomposition is clarified.

THEOREM 6. *Let, for an $n \times n$ matrix A and for a decomposition $N = N_1 \cup N_2 \cup \cdots \cup N_p$, all principal submatrices*

$$A\left[\bigcup_{j=1}^{k} N_j \middle| \bigcup_{j=1}^{k} N_j\right], \qquad k = 1, \dots, p,$$

be nonsingular, so that a block LU decomposition of A exists, where L has zero blocks in all $N_j \times N_k$ positions for $j < k$ and U has zero blocks in all $N_j \times N_k$ positions for $j > k$. Then, in the notation (8),

$$r(S_{ub}; U) = r(S_{ub}; A). \tag{11}$$

In other words, U has the same strictly upper-block-triangular rank as A (and as A^{-1}, by Theorem 3).

Proof. Since $A = LU$,

$$A\left[\bigcup_{i=1}^{k} N_i \middle| N \setminus \bigcup_{i=1}^{k} N_i\right] = L\left[\bigcup_{i=1}^{k} N_i \middle| \bigcup_{i=1}^{k} N_i\right] U\left[\bigcup_{i=1}^{k} N_i \middle| N \setminus \bigcup_{i=1}^{k} N_i\right],$$

which implies

$$\operatorname{rank} A\left[\bigcup_{i=1}^{k} N_i \middle| N \setminus \bigcup_{i=1}^{k} N_i\right] = \operatorname{rank} U\left[\bigcup_{i=1}^{k} N_i \middle| N \setminus \bigcup_{i=1}^{k} N_i\right],$$

$k = 1, \dots, p$.

By (6), (11) follows. ∎

3. APPLICATIONS

In this concluding section, we shall first generalize the structure rank as follows: We assume that in $M \times N$, two structures S_1, S_2 are given such that $S_1 \subseteqq S_2$, $S_1 \neq S_2$. If A is an $m \times n$ matrix, the *difference* $S_2 \ominus S_1$*-rank* of A is the maximum order of a nonsingular submatrix $A[\alpha|\beta]$ for which

$$S_1 \subseteqq \alpha \times \beta \subseteqq S_2.$$

We denote it as $r(S_2 \ominus S_1; A)$, so that

$$r(S_2 \ominus S_1; A) = \max\{\text{rank } A[\alpha|\beta] \,|\, \alpha \subseteqq M,\ \beta \subseteqq N,\ S_1 \subseteqq \alpha \times \beta \subseteqq S_2\}.$$

In this notation, we prove:

THEOREM 7. *Let* $N = N_1 \cup \cdots \cup N_p$, $p \geqslant 2$, *be a decomposition of* N. *Let* A *be an* $n \times n$ *matrix for which* $A[N_1|N_1]$ *is invertible. Then, for*

$$S_1 = N_1 \times N_1, \qquad S_2 = N \times N \setminus \bigcup_{i=1}^{p} (N_i \times N_i),$$

and

$$\hat{S} = \bigcup_{i=2}^{p} N_i \times \bigcup_{i=2}^{p} N_i \setminus \bigcup_{i=2}^{p} (N_i \times N_i)$$

(in $(N \setminus N_1) \times (N \setminus N_1)$*), one has*

$$r(S_2 \ominus S_1; A) = r\left(\hat{S};\ A^{-1}\left[\bigcup_{i=2}^{p} N_i \,\middle|\, \bigcup_{i=2}^{p} N_i\right]\right) + |N_1|. \tag{12}$$

Proof. For notational convenience, we can assume that N_1 consists of the first n_1 indices, N_2 of the next n_2 consecutive indices, ..., N_p of the last n_p indices and $\sum_{k=1}^{p} n_k = n$. We can then write A in the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is $n_1 \times n_1$, A_{22} $(n_2 + \cdots + n_p) \times (n_2 + \cdots + n_p)$. Let the inverse conformally partitioned be denoted as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

As is well known [3], the Schur complement of A_{11} in A is

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

and satisfies

$$(A/A_{11})^{-1} = B_{22}.$$

By Theorem 2 on off-block-diagonal ranks,

$$r(\hat{S};\, B_{22}) = r(\hat{S};\, A/A_{11}),$$

which is, however, equal to

$$r(S_2 \ominus S_1;\, A) - n_1,$$

since

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \begin{pmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{pmatrix}.$$

This proves (12) in the original notation. ■

Let us add an observation on orthogonal and unitary matrices.

THEOREM 8. *Let C be a unitary or orthogonal matrix. Assign to every $p \times q$ submatrix C_0 of C the number $k(C_0) = \frac{1}{2}(p+q) - \operatorname{rank} C_0$. Then, whenever C_1 and C_2 are complementary submatrices of C (the rows as well as the columns are complementary), one has $k(C_1) = k(C_2)$.*

Proof. Follows easily from (2) and the fact that $C^{-1} = C^*$ (or C^T). ■

COROLLARY 9. *In a unitary or orthogonal $n \times n$ matrix, the ranks of complementary $p \times q$ submatrices with $p + q = n$ coincide.*

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